Adapting Nikoloulopoulos adaptation of Johnson and Krotz

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Nikoloulopoulos (2020) describes using an approximation from Johnson and Kotz (1992) for his R function.

For this the derivatives of the standard normal pdf need to be calculated. The first derivative is:

$$\phi'(z) = -z \times \phi(z)$$

If we substitute:

$$f(z) = -z$$

Rewrite using substitution:

$$\phi'(z) = f(z) \times \phi(z)$$

Then we can apply the product rule:

$$\phi''(z) = f(z) \times \phi'(z) + f'(z) \times \phi(z)$$

With:

$$f'(z) = -1$$

Which gives:

$$\phi''(z) = -z \times -z \times \phi(z) + -1 \times \phi(z)$$

$$= z^2 \times \phi(z) - \phi(z)$$

$$= (z^2 - 1) \times \phi(z)$$

For the third derivative, we use:

$$f(z) = z^2 - 1$$

With:

$$f'(z) = 2 \times z$$

Rewrite using substitution:

$$\phi'''(z) = \phi'\left(\phi''(z)\right) = \phi'\left(f(z) \times \phi(z)\right)$$

Then apply product rule:

$$\phi''(z) = f(z) \times \phi'(z) + f'(z) \times \phi(z)$$

$$= (z^2 - 1) \times -z \times \phi(z) + 2 \times z \times \phi(z)$$

$$= (-z^3 + z) \times \phi(z) + 2 \times z \times \phi(z)$$

$$= (-z^3 + 3z) \times \phi(z)$$

For the fourth derivative, we use:

Substitute:

$$f(z) = -z^3 + 3z$$

With:

$$f'(z) = -3z^2 + 3$$

Rewrite using substitution:

$$\phi''''(z) = \phi'\big(\phi'''(z)\big) = \phi'\big(f(z) \times \phi(z)\big)$$

Then apply product rule:

$$\phi''(z) = f(z)\phi'(z) + f'(z)\phi(z)$$

$$= (-z^3 + 3z) \times -z\phi(z) + (-3z^2 + 3)\phi(z)$$

$$= (z^4 - 3z^2)\phi(z) + (-3z^2 + 3) \times \phi(z)$$

$$= (z^4 - 6z^2 + 3) \times \phi(z)$$

For the fifth derivative, we use:

Substitute:

$$f(z) = z^4 - 6z^2 + 3$$

With:

$$f'^{(z)} = 4z^3 - 12z$$

Rewrite using substitution:

$$\phi'''''(z) = \phi'\big(\phi''''(z)\big) = \phi'\big(f(z) \times \phi(z)\big)$$

Then apply product rule:

$$\phi''(z) = f(z)\phi'(z) + f'(z)\phi(z)$$

$$= (z^4 - 6z^2 + 3) \times -z\phi(z) + (4z^3 - 12z)\phi(z)$$

$$= (-z^5 + 6z^3 - 3z)\phi(z) + (4z^3 - 12z) \times \phi(z)$$

$$= (-z^5 + 10z^3 - 15z) \times \phi(z)$$

To make this process less tedious, let’s make it an iterative process.

First notice that:

$$\phi^x(z) = (-1)^x \times g_x \times \phi(z)$$

Where $\phi^x(z)$ is the xth derivative of $\phi(z)$. And we define iterative:

$$g_x = z^x + c_1 z^{x-2} + c_2 z^{x-4} + \cdots +$$

$$g_{x+1} = z^{x+1} + (c_1 + x)z^{x-1} + \big(c_2 + c_1(x-2)\big)z^{x-3} + \cdots +$$

For the coefficient in general:

$$c_{x-1} = c_{x-2} + c_x \times (x-1)$$

Let's try out a few values:

$$g_1 = z^1 = z$$

$$g_2 = g_{1+1} = z^{1+1} + (c_1 + x)z^{1-1} = z^2 + (0+1)z^0 = z^2 + 1$$

$$g_3 = g_{2+1} = z^{2+1} + (c_1 + 2)z^{2-1} = z^3 + (1+2)z = z^3 + 3z$$

$$g_4 = g_{3+1} = z^{3+1} + (c_1 + 3)z^{3-1} + \big(c_2 + c_1(3-2)\big)z^{3-3} = z^4 + (c_1+3)z^2 + (c_2 + c_1)$$
$$= z^4 + (3+3)z^2 + (0+3)z = z^4 + 6z^2 + 3$$

$$g_5 = g_{4+1} = z^{4+1} + (c_1 + 4)z^{4-1} + \big(c_2 + c_1(4-2)\big)z^{4-3} = z^5 + (c_1+4)z^3 + (c_2 + 2c_1)z$$
$$= z^5 + (6+4)z^3 + (3 + 2\times 6)z = z^5 + 10z^3 + 15z$$

References

Kotz, S., & Johnson, N. L. (Eds.). (1992). *Breakthroughs in Statistics Volume 1: Foundations and Basic Theory*. Springer.

Nikoloulopoulos, A. (2020, March 23). *Approximation of bivariate standard normal distribution*. R Documentation. https://www.rdocumentation.org/packages/weightedScores/versions/0.9.5.3/topics/approxbvncdf