1 Re-written from Tsay and Ke (2021)

If $a > 0$ and $a \times q + b \geq 0$

$$bsncdf(p,q,p) \approx \phi(q) + \phi\left(\frac{b}{a}\right) - 1 + f_1\left(f_3\big(1 - \phi(v_5)\big) - f_4(\phi(v_7) + \phi(v_6) - 1)\right)$$

If $a > 0$ and $a \times q + b < 0$

$$bsncdf(p,q,p) \approx f_1 f_3 \phi(v_8)$$

If $a = 0$

$$bsncdf(p,q,p) = \phi(p)\phi(q)$$

If $a < 0$ and $a \times q + b \geq 0$

$$bsncdf(p,q,p) \approx \phi(q) - f_1 f_4 \phi(v_7)$$

If $a < 0$ and $a \times q + b < 0$

$$bsncdf(p,q,p) \approx 1 - \phi\left(\frac{b}{a}\right) + f_1\left(f_3(\phi(v_8) + \phi(v_5) - 1) - f_4\big(1 - \phi(v_6)\big)\right)$$

With:

$$a = \frac{-\rho}{\sqrt{1-\rho^2}}, b = \frac{p}{\sqrt{1-\rho^2}}$$

$$c_1 = -1.09500814703333$$

$$c_2 = -0.75651138383854$$

$$f_1 = \frac{1}{2\sqrt{1 - a^2 c_2}}$$

$$f_2 = e^{f_1^2 \times (a^2 c_1^2 + 2b^2 c_2)}$$

$$f_3 = f_2 \times e^{-f_1^2 \times 2\sqrt{2} b c_1}$$

$$f_4 = f_2 \times e^{f_1^2 \times 2\sqrt{2} b c_1}$$

$$v_5 = \frac{f_1}{a} \times \left(2b - \sqrt{2} a^2 c_1\right)$$

$$v_6 = \frac{f_1}{a} \times \left(2b + \sqrt{2} a^2 c_1\right)$$

$$v_7 = f_1 \times \left(2q - 2a^2 c_2 q - 2abc_2 - \sqrt{2} a c_1\right)$$

$$v_8 = f_1\left(2q - 2a^2 c_2 q - 2abc_2 + \sqrt{2} a c_1\right)$$

2 R-code

```
bsncdf <- function(p, q, rho){
  c1 = -1.09500814703333
  c2 = -0.75651138383854
  a = -rho/sqrt(1-rho**2)
  if(a==0){pVal = pnorm(p)*pnorm(q)}
  else{
    b = p/sqrt(1-rho**2)
    f1 = 1/(2*sqrt(1-a**2*c2))
    f2 = exp(f1**2 * (a**2*c1**2 + 2*b**2*c2))
    cond1 = a > 0
    if(cond1){sgn = 1}
    else{sgn = -1}
    f3 = f2*exp(-sgn*f1**2*2*sqrt(2)*b*c1)
    cond2 = a*q + b >= 0
    if(cond1 == cond2){
      f4 = f2*exp(sgn*f1**2*2*sqrt(2)*b*c1)
      v5 = f1/a*(2*b-sgn*sqrt(2)*a**2*c1)
      v6 = f1/a*(2*b+sgn*sqrt(2)*a**2*c1)
      v8 = f1*(2*q-2*a**2*c2*q-2*a*b*c2-sgn*sqrt(2)*a*c1)
    } else{
      v8 = f1*(2*q-2*a**2*c2*q-2*a*b*c2+sgn*sqrt(2)*a*c1)
    }
    if(cond1 && cond2){
      pVal = pnorm(q) + pnorm(b/a) - 1 + f1*(f3*(1-pnorm(v5)) - f4*(pnorm(v8)+pnorm(v6)-1))
    } else if(cond1 && !cond2){
      pVal = f1*f3*pnorm(v8)
    } else if(!cond1 && cond2){
      pVal = pnorm(q) - f1*f3*pnorm(v8)
    } else{
      pVal = 1 - pnorm(b/a) + f1*(f4*(pnorm(v8)+pnorm(v6)-1) - f3*(1-pnorm(v5)))
    }
  }
  return(pVal)
}

for (i in c(-0.99, -0.5, -0.1, 0.1, 0.5, 0.99)) {
  print(bsncdf(1, 2, i)*10**5)

}
```

3 Proof

Original:

If $a > 0$ and $a \times q + b \geq 0$

$$bsncdf(p,q,p) \approx \frac{1}{2} \times \left(erf\left(\frac{q}{\sqrt{2}}\right) + erf\left(\frac{b}{\sqrt{2}a}\right)\right) + \frac{1}{4\sqrt{1-a^2c_2}} e^{\frac{a^2c_1^2-\sqrt{2}bc_1+2b^2c_2}{4(1-a^2c_2)}} \times \left(1 - erf\left(\frac{\sqrt{2}b - a^2c_1}{2a\sqrt{1-a^2c_2}}\right)\right) - \frac{1}{4\sqrt{1-a^2c_2}} e^{\frac{a^2c_1^2+\sqrt{2}bc_1+2b^2c_2}{4(1-a^2c_2)}} \times \left(erf\left(\frac{\sqrt{2}q - \sqrt{2}a^2c_2q - \sqrt{2}abc_2 - ac_1}{2\sqrt{1-a^2c_2}}\right) + erf\left(\frac{\sqrt{2}b + a^2c_1}{2a \times \sqrt{1-a^2c_2}}\right)\right)$$

Substitutions:

Let's use the following substitutions:

$$f_1 = \frac{1}{2\sqrt{1-a^2c_2}}$$

$$f_2 = e^{\frac{a^2c_1^2+2b^2c_2}{4(1-a^2c_2)}} = e^{f_1^2 \times (a^2 \times c_1^2 + 2 \times b^2 \times c_2)}$$

$$f_3 = e^{\frac{a^2c_1^2-\sqrt{2}bc_1+2b^2c_2}{4(1-a^2c_2)}} = f_2 \times e^{-f_1^2 \times 2\sqrt{2}bc_1}$$

$$f_4 = e^{\frac{a^2c_1^2+\sqrt{2}bc_1+2b^2c_2}{4(1-a^2c_2)}} = f_2 \times e^{f_1^2 \times 2 \times \sqrt{2} \times b \times c_1}$$

$$f_5 = \frac{\sqrt{2} \times b - a^2 \times c_1}{2 \times a \times \sqrt{1 - a^2 \times c_2}} = \frac{f_1}{a} \times \left(\sqrt{2} \times b - a^2 \times c_1\right)$$

$$f_6 = \frac{\sqrt{2} \times b + a^2 \times c_1}{2 \times a \times \sqrt{1 - a^2 \times c_2}} = \frac{f_1}{a} \times \left(\sqrt{2} \times b + a^2 \times c_1\right)$$

$$f_7 = f_1 \times \left(\sqrt{2} \times q - \sqrt{2} \times a^2 \times c_2 \times q - \sqrt{2} \times a \times b \times c_2 - a \times c_1\right)$$

New:

If $a > 0$ and $a \times q + b \geq 0$

$$bsncdf(p,q,p) \approx \frac{1}{2} \times \left(erf\left(\frac{q}{\sqrt{2}}\right) + erf\left(\frac{b}{\sqrt{2}a}\right)\right) + \frac{f_1}{2} f_3 \times \left(1 - erf(f_5)\right) - \frac{f_1}{2} f_4 \times \left(erf(f_7) + erf(f_6)\right)$$

Original

If $a > 0$ and $a \times q + b < 0$

$$bsncdf(p,q,p) \approx \frac{1}{4\sqrt{1-a^2c_2}} e^{\frac{a^2c_1^2 - \sqrt{2}bc_1 + 2b^2c_2}{4(1-a^2c_2)}} \times \left(1 + erf\left(\frac{\sqrt{2}q - \sqrt{2}a^2c_2q - \sqrt{2}abc_2 + ac_1}{2\sqrt{1-a^2c_2}}\right)\right)$$

Substitution

Using our previous substitutions and:

$$f_8 = f_1 \times \left(\sqrt{2} \times q - \sqrt{2} \times a^2 \times c_2 \times q - \sqrt{2} \times a \times b \times c_2 + a \times c_1\right)$$

Re-written

This can now be rewritten to:

$$bsncdf(p,q,p) \approx \frac{f_1}{2} f_3 \times \left(1 + erf(f_8)\right)$$

Original

If $a < 0$ and $a \times q + b \geq 0$

$$bsncdf(p,q,p) \approx \frac{1}{2} + \frac{1}{2} erf\left(\frac{q}{\sqrt{2}}\right) - \frac{1}{4\sqrt{1-a^2c_2}} e^{\frac{a^2c_1^2 + \sqrt{2}bc_1 + 2b^2c_2}{4(1-a^2c_2)}} \times \left(1 + erf\left(\frac{\sqrt{2}q - \sqrt{2}a^2c_2q - \sqrt{2}abc_2 - ac_1}{2\sqrt{1-a^2c_2}}\right)\right)$$

Substitution

Using earlier substitutions, we can rewrite this to:

$$bsncdf(p,q,p) \approx \frac{1}{2}\left(1 + erf\left(\frac{q}{\sqrt{2}}\right) - f_1 \times f_4 \times \left(1 + erf(f_7)\right)\right)$$

Original

If $a < 0$ and $a \times q + b < 0$

$$\begin{aligned} bsncdf(p,q,p) \approx{} & \frac{1}{2} - \frac{1}{2} erf\left(\frac{b}{\sqrt{2}a}\right) \\ & - \frac{1}{4\sqrt{1-a^2c_2}} \times e^{\frac{a^2c_1^2 + \sqrt{2}bc_1 + 2b^2c_2}{4(1-a^2c_2)}} \times \left(1 - erf\left(\frac{\sqrt{2}b + a^2c_1}{2a \times \sqrt{1-a^2c_2}}\right)\right) \\ & + \frac{1}{4\sqrt{1-a^2{c_2}_1}} e^{\frac{a^2c_1^2 - \sqrt{2}bc_1 + 2b^2c_2}{4(1-a^2c_2)}} \\ & \times \left(erf\left(\frac{\sqrt{2}q - \sqrt{2}a^2c_2q - \sqrt{2}abc_2 + ac_1}{2\sqrt{1-a^2c_2}}\right) + erf\left(\frac{\sqrt{2}b - a^2c_1}{2a \times \sqrt{1-a^2c_2}}\right)\right) \end{aligned}$$

Substitution

$$bsncdf(p,q,p) \approx \frac{1}{2}\left(1 - erf\left(\frac{b}{\sqrt{2} \times a}\right) - f_1 \times f_4 \times \left(1 - erf(f_6)\right) + f_1 \times f_3 \times \left(erf(f_8) + erf(f_5)\right)\right)$$

Converting to normal distribution

Next, we want to get to the cumulative standard normal distribution, instead of the erf. Since:

$$erf(x) = 2\Phi\left(x\sqrt{2}\right) - 1$$

We can substitute this in each of them.

$$bsncdf(p,q,p) \approx \frac{1}{2} \times \left(erf\left(\frac{q}{\sqrt{2}}\right) + erf\left(\frac{b}{\sqrt{2} \times a}\right)\right) + \frac{f_1}{2} \times f_3 \times \left(1 - erf(f_5)\right) - \frac{f_1}{2} \times f_4 \times \left(erf(f_7) + erf(f_6)\right)$$

$$\approx \frac{1}{2} \times \left(2 \times \phi(q) - 1 + 2 \times \phi\left(\frac{b}{a}\right) - 1\right) + \frac{f_1}{2} \times f_3 \times \left(1 - 2 \times \phi\left(\sqrt{2} \times f_5\right) + 1\right) - \frac{f_1}{2} \times f_4 \times \left(2 \times \phi\left(\sqrt{2} \times f_7\right) - 1 + 2 \times \phi\left(\sqrt{2} \times f_6\right) - 1\right)$$

$$\approx \phi(q) + \phi\left(\frac{b}{a}\right) - 1 + f_1 \times \left(f_3 \times \left(1 - \phi\left(\sqrt{2} \times f_5\right)\right) - f_4 \times \left(\phi\left(\sqrt{2} \times f_7\right) + \phi\left(\sqrt{2} \times f_6\right) - 1\right)\right)$$

$$\approx \phi(q) + \phi\left(\frac{b}{a}\right) - 1 + f_1 \times \left(f_3 \times \left(1 - \phi(v_5)\right) - f_4 \times \left(\phi(v_7) + \phi(v_6) - 1\right)\right)$$

If $a > 0$ and $a \times q + b < 0$

$$bsncdf(p,q,p) \approx \frac{f_1}{2} \times f_3 \times \left(1 + erf(f_8)\right)$$

$$\approx \frac{f_1}{2} \times f_3 \times \left(1 + 2 \times \phi\left(\sqrt{2} \times f_8\right) - 1\right)$$

$$\approx f_1 \times f_3 \times \phi(v_8)$$

If $a < 0$ and $a \times q + b \geq 0$

$$bsncdf(p,q,p) \approx \frac{1}{2}\left(1 + erf\left(\frac{q}{\sqrt{2}}\right) - f_1 \times f_4 \times \left(1 + erf(f_7)\right)\right)$$

$$\approx \frac{1}{2}\left(1 + 2 \times \phi\left(\sqrt{2} \times \frac{q}{\sqrt{2}}\right) - 1 - f_1 \times f_4 \times \left(1 + 2 \times \phi\left(\sqrt{2} \times f_7\right) - 1\right)\right)$$

$$\approx \frac{1}{2}\left(2 \times \phi\left(\sqrt{2} \times \frac{q}{\sqrt{2}}\right) - f_1 \times f_4 \times 2 \times \phi\left(\sqrt{2} \times f_7\right)\right)$$

$$\approx \phi(q) - f_1 \times f_4 \times \phi(v_7)$$

If $a < 0$ and $a \times q + b < 0$

$$bsncdf(p,q,p) \approx \frac{1}{2}\left(1 - erf\left(\frac{b}{\sqrt{2} \times a}\right) - f_1 \times f_4 \times \big(1 - erf(f_6)\big) + f_1 \times f_3 \times \big(erf(f_8) + erf(f_5)\big)\right)$$

$$\approx \frac{1}{2}\left(1 - 2 \times \phi\left(\sqrt{2} \times \frac{b}{\sqrt{2} \times a}\right) + 1 - f_1 \times f_4 \times \big(1 - 2 \times \phi(\sqrt{2} \times f_6) + 1\big) + f_1 \times f_3 \times \big(2 \times \phi(\sqrt{2} \times f_8) - 1 + 2 \times \phi(\sqrt{2} \times f_5) - 1\big)\right)$$

$$\approx \frac{1}{2}\left(2 - 2 \times \phi\left(\frac{b}{a}\right) - f_1 \times f_4 \times \big(2 - 2 \times \phi(v_6)\big) + 2 \times f_1 \times f_3 \times (\phi(v_8) + \phi(v_5) - 1)\right)$$

$$\approx \frac{1}{2}\left(2 - 2 \times \phi\left(\frac{b}{a}\right) - 2 \times f_1 \times f_4 \times \big(1 - \phi(v_6)\big) + 2 \times f_1 \times f_3 \times (\phi(v_8) + \phi(v_5) - 1)\right)$$

$$\approx 1 - \phi\left(\frac{b}{a}\right) - f_1 \times f_4 \times \big(1 - \phi(v_6)\big) + f_1 \times f_3 \times (\phi(v_8) + \phi(v_5) - 1)$$

$$\approx 1 - \phi\left(\frac{b}{a}\right) + f_1 \times f_3 \times (\phi(v_8) + \phi(v_5) - 1) - f_1 \times f_4 \times \big(1 - \phi(v_6)\big)$$

$$\approx 1 - \phi\left(\frac{b}{a}\right) + f_1 \times \Big(f_3 \times (\phi(v_8) + \phi(v_5) - 1) - f_4 \times \big(1 - \phi(v_6)\big)\Big)$$

With:

$$v_5 = \sqrt{2} \times f_5 = \sqrt{2} \times \frac{f_1}{a} \times \big(\sqrt{2} \times b - a^2 \times c_1\big) = \frac{f_1}{a} \times \big(2 \times b - \sqrt{2} \times a^2 \times c_1\big)$$

$$v_6 = \sqrt{2} \times f_6 = \sqrt{2} \times \frac{f_1}{a} \times \big(\sqrt{2} \times b + a^2 \times c_1\big) = \frac{f_1}{a} \times \big(2 \times b + \sqrt{2} \times a^2 \times c_1\big)$$

$$v_7 = \sqrt{2} \times f_7 = \sqrt{2} \times f_1 \times \big(\sqrt{2} \times q - \sqrt{2} \times a^2 \times c_2 \times q - \sqrt{2} \times a \times b \times c_2 - a \times c_1\big) = f_1 \times \big(2 \times q - 2 \times a^2 \times c_2 \times q - 2 \times a \times b \times c_2 - \sqrt{2} \times a \times c_1\big)$$

$$v_8 = \sqrt{2} \times f_8 = \sqrt{2} \times f_1 \times \big(\sqrt{2} \times q - \sqrt{2} \times a^2 \times c_2 \times q - \sqrt{2} \times a \times b \times c_2 + a \times c_1\big) = f_1 \times \big(2 \times q - 2 \times a^2 \times c_2 \times q - 2 \times a \times b \times c_2 + \sqrt{2} \times a \times c_1\big)$$